Reg. No. | | | | | | | | | |

MTH 403

First Semester M.Sc. Degree Examination, December 2018/January 2019
MATHEMATICS
Real Analysis – I
Choice Based Credit System – New Syllabus

Time : 3 Hours Max. Marks : 70

Note : *1) Answer* ***any five full*** *questions.*
2) Answer to ***each full*** *question shall* ***not*** *exceed* ***eight*** *pages of the answer book. No additional sheets will be provided for answering.*
3) ***Use*** *of scientific calculator is* ***permitted****.*

1. a) State the least upper bound property. Prove that an ordered set having the least upper bound property also has the greatest lower bound property.
   b) For every real $x > 0$ and every integer $n > 0$ prove that there is only one real y such that $y^n = x$.
   c) Let A be a nonempty set of real numbers which is bounded below. Then prove that $\inf A = -\sup(-A)$. **(4+6+4)**

2. a) Define a countable set. Prove that the set $\mathbb{Z}$ of all integers is countable.
   b) Prove that the set of all sequences whose elements are the digits 0 and 1 is uncountable.
   c) Define an algebraic number. Prove that the set of all algebraic numbers is countable. **(5+5+4)**

3. a) Define a metric space. Prove that $d(x, y) = \dfrac{|x-y|}{1+|x-y|}$, is a metric on $\mathbb{R}$.
   b) Prove that every neighbourhood of a point in a metric space is open.
   c) Show that an arbitrary intersection of open sets need not be open in a metric space. **(5+5+4)**

4. a) Define a compact space. Prove that compact subsets of metric spaces are closed.
   b) Prove that every nonempty perfect set in $\mathbb{R}^k$ is uncountable.
   c) Prove that a subset E of $\mathbb{R}$ is connected if it is an interval. **(5+7+2)**

5. a) Define a sequence. Prove that every convergent sequence in a metric space is bounded. How about the converse ? Justify.

   b) Prove that every bounded sequence in $\mathbb{R}^k$ contains a convergent subsequence.

   c) Prove the following :

      i) $\lim_{n\to\infty} \sqrt[n]{n} = 1$

      ii) If $|x| < 1$, then $\lim_{n\to\infty} x^n = 0$. **(4+4+6)**

6. a) Suppose $a_n \geq a_{n+1}$ and $a_n \geq 0$ for n = 1, 2,... Then prove that the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the series $\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges.

   b) Define the number e. Prove that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$.

   c) Investigate the behaviour of the following series :

      i) $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{n}$

      ii) $\frac{1}{2}+\frac{1}{3}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{2^3}+\frac{1}{3^3}+\ldots$. **(6+4+4)**

7. a) Prove that a mapping f of a metric space X into a metric space Y is continuous on X if and only if $f^{-1}(V)$ is open in X for every open set V in Y.

   b) Prove that a continuous mapping of a compact metric space X into a metric space Y is uniformly continuous.

   c) If f is a continuous real function on a metric space X, show that the set, $\{x | f(x) = 0\}$ is closed. **(6+6+2)**

8. a) If f and g are continuous real functions on [a, b] which are differentiable on (a, b), then prove that there is a point $x \in (a, b)$ such that $[f(b) - f(a)]\, g'(x) = [g(b) - g(a)]f'(x)$.

   b) State and prove the Taylor's theorem.

   c) Suppose f is defined in a neighbourhood of x and suppose $f''(x)$ exists. Show that $\lim_{h\to 0}\frac{f(x+h)+f(x-h)-2f(x)}{h^2} = f''(x)$. **(6+6+2)**